THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3070 (Second Term, 2016–2017)
Introduction to Topology
Exercise 2 Open and Closed Sets

Remarks

Many of these exercises are adopted from the textbooks (Davis or Munkres). You are suggested to work more from the textbooks or other relevant books.

1. Do the exercises mentioned in lectures or in lecture notes.

2. Show that for any $A \subset X$, $\overline{A}$ and $\operatorname{Frt}(A)$, but not necessarily A', are closed.

3. Let $\mathbb{R}$ be given the cofinite topology. What are $\mathring{\mathbb{Q}}$, $\overline{\mathbb{Q}}$ and $\operatorname{Frt}(\mathbb{Q})$?

4. The closure of A can be defined as $\overline{A} = \cap \{ A \subset F : X \setminus F \in \mathfrak{T} \}$ or
$$\overline{A} = \{ x \in X : \text{ for all } U \in \mathfrak{T} \text{ with } x \in U,\, U \cap A \neq \emptyset \} .$$
Show that these two are equivalent. Again, U can be replaced with a neighborhood N of x.

5. Check if the following statements are true for a general topological space.
   (a) $\overline{A} = X \setminus \operatorname{Int}(X \setminus A)$
   (b) $\operatorname{Int}(A) = X \setminus \overline{(X \setminus A)}$
   (c) $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$ and $\operatorname{Int}(A \cup B) \supset \mathring{A} \cup \mathring{B}$ but not necessarily equal.
   (d) What if we change the $\cap$ to $\cup$ and $\cup$ to $\cap$ above?
   (e) $\operatorname{Int}(A) \cup \operatorname{Frt}(A) = \overline{A}$.

6. Recall that for $Y \subset X$, the *induced topology* or *relative topology* on Y is
$$\mathfrak{T}|_Y = \{ G \cap Y : G \in \mathfrak{T} \} .$$
Let $A \subset Y \subset X$. What are the relation between $\operatorname{Int}_Y(A)$ and $\operatorname{Int}_X(A)$; $\operatorname{Cl}_Y(A)$ and $\operatorname{Cl}_X(A)$; and $\operatorname{Frt}_Y(A)$ and $\operatorname{Frt}_X(A)$? Further deduce the results for the special situation that either A is open or closed in X.

7. Show that every finite subset in a metric space (X, d) is a closed set.
   (a) Give an example of a countable subset in a metric space that is not a closed set.
   (b) Give an example of a countable subset in a metric space that is still a closed set.

(c) Cook up other examples by changing the above (this is a good attitude of learning topology, or even any mathematics).

8. On a metric space (X, d), is it true that $\overline{B(x,r)} = \{ y \in X : d(x,y) \leq r \}$? Also, show that

$$\overline{A} = \{ x \in X : d(x, A) = 0 \}, \qquad \text{where } d(x, A) :\overset{\text{def}}{=\!=} \inf \{ d(x,a) : a \in A \}.$$

9. For a general topological space $(X, \mathfrak{T})$,

   (a) Is there an example of $(X, \mathfrak{T})$ such that $\mathrm{Frt}(A) \neq \overline{A} \setminus \mathrm{Int}(A)$?

   (b) For an open set U, is it true that $U = \mathrm{Int}(\mathrm{Cl}(U))$?

   (c) Is it true that $\overline{A \setminus B} = \overline{A} \setminus \mathrm{Int}\, B$?

10. Compare $\mathrm{Int}(\mathrm{Cl}(A))$ and $\mathrm{Cl}(\mathrm{Int}(A))$. Are they equal or one is a subset of another?

11. Think about the typical closed sets (or closure) for the order topology and $\mathfrak{T}_{cf0}$ given in HW01.

12. Google "Kuratowski 14 sets" and understand what it says.